Intelligent Methods in Engineering Sciences 4(2): 54-65, 2025

INTELLIGENT METHODS IN ENGINEERING SCIENCES

https://www.imiens.org

International Open Access

August, 2025

e-ISSN 2979-9236

Research Article

https://doi.org/10.58190/imiens.2025.127

A Fixed-Grid Rdtm-Based Computational Strategy for Nonlinear Partial Differential Equations

Sema Servi [a,*], ***Galip Oturanç*** [b]

[a] *Department of Computer Engineering, Faculty of Technology, Selçuk University, Konya, Türkiye*
[b] *Department of Mathematics, Faculty of Science, Karamanoğlu Mehmetbey University, Karaman, Türkiye*

ARTICLE INFO

Article history:
Received 31 July 2025
Accepted 24 August 2025

Keywords:
Partial Differential Equations
Reduced Differential Transform Method
Fixed-Grid Size
Intelligent Systems
Klein-Gordon equation

ABSTRACT

In this study, a fixed-grid version of the Reduced Differential Transform Method (RDTM) is systematically implemented to obtain approximate solutions of linear and nonlinear partial differential equations. In this method, the solution range is divided into equal subregions and the fixed-grid algorithm is integrated into the RDTM framework. This approach provides an efficient and orderly computational process for solving complex partial differential equations. The effectiveness of the proposed method is demonstrated on the homogeneous Klein–Gordon equation (a representative hyperbolic equation) and the nonlinear Klein–Gordon equation, and the obtained approximate solutions are compared with known analytical solutions with high accuracy and consistency. Furthermore, the proposed fixed-grid RDTM (FGS-RDTM) framework offers potential integration with intelligent systems where accurate and efficient numerical solvers are required for modeling, control, and learning in dynamic environments. These results confirm the reliability and practical usefulness of the new method in addressing nonlinear partial differential equations in the context of intelligent computational systems.

1. INTRODUCTION

Partial differential equations (PDEs) are central to the mathematical modeling of physical, chemical, and mechanical processes in many disciplines, ranging from the natural sciences to engineering. They are indispensable for describing the behavior of complex processes, particularly in multidisciplinary fields such as fluid mechanics, thermodynamics, heat transfer, electromagnetism, mathematical physics, and artificial intelligence-based systems modeling. In recent years, advances in both mathematical methods and computational technologies have enabled the development of efficient solution techniques for analytical and approximate solutions of these equations.

In recent years, with the advances in theoretical mathematics and the development of numerical computing technologies, many effective methods for the analytical, semi-analytical and numerical solutions of partial differential equations (PDEs) have been introduced into the literature. In this context, iterative approaches such as the Variational Iteration Method (VIM) [1-3], the Homotopy Analysis Method (HAM) [4] and the Homotopy Perturbation Method (HPM) [3, 5, 6] stand out among the widely used methods, especially in the solution of linear and nonlinear PDEs. In addition, the Differential Transform Method (DTM) [7, 8] and its improved version, the Reduced Differential Transform Method (RDTM), are frequently preferred, especially for their ease of use and speed of convergence. RDTM was introduced in 2009 as an improved version of DTM; the approach provides more efficient results compared to DTM, especially by reducing the number of iterations and the computational cost[9]. The advantages of this methodology in nonlinear PDEs have been clearly demonstrated on problems such as the homogeneous heat equation, the Burgers equation, the KDV equation, and the Klein–Gordon equation [10-21]

The Adomian Decomposition Method (ADM) [22-24] is a semi-analytical technique that decomposes nonlinear operator equations into a set of functions and allows for the systematic treatment of nonlinear terms in differential equations [25]; the Elzaki and Laplace Transform Methods are used to solve initial value problems with integral transform techniques [26, 27]. In addition, the Sine-Cosine Method [28], the Taylors Series Method [28], the Galerkin and Petrov–Galerkin Methods, Spectral Methods with Chebyshev Polynomials [29], the Variational Iterative Method (VIM), and Hybrid Methods, including the VIM-HPM hybrid and the Adomian-Laplace Combination, have provided effective solution strategies for both linear and

* **Corresponding Author:** semaservi@selcuk.edu.tr

nonlinear KDPs in recent years [27, 29, 30]. More recently, fractional and integral-based approaches such as VIE RDTM (RDTM extended with Volterra Integral Equations) have been developed [30]; they reduce the error rate and expand the area of convergence in applications ranging from the Black-Scholes equation to binary heat wave-type differential equations [31]. Additionally, studies published in the 2022–2023 period emphasized the hybrid applications of RDTM with the Padé approach in more complex systems such as two-dimensional convection-diffusion equations [30] and its high accuracy [32].

The fixed grid size method, also called RDTM, was first proposed by Servi in his doctoral studies and has been successfully applied to problems such as the homogeneous heat equation [21], the Burgers equation[21], the Newell–Whitehead–Segel (NWS) equation [33], the telegraph equations [34], the wave equations [35], the Sine-Gordon equations [34], and Gaousat [36]. In this method, by dividing the solution domain into fixed-range subregions, error control is facilitated and the convergence rate is significantly increased. This grid-based algorithm was previously used by Ming-Jyi Jang, Chieh-Li Chen, and Yung-Chin Liy to solve linear and nonlinear initial value problems with DTM, and positive results were obtained [37].

2. MATERIAL AND METHODS

2.1. Reduced Differential Transform Method with Fixed Grid Size Solution

The basic concepts and definitions of RDTM (Reduced Differential Transform Method) have been discussed comprehensively in the works [9, 21]

Definition

If the function $u(x,t)$ is analytically and continuously differentiable on a certain interval with respect to both time t and space x variables, then the spectrum function $U_k(x)$ in dimension t is obtained as the transformed form of that function.

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} \tag{2.1}$$

In this study, the lowercase letters $u(x,t)$ represent the original function, while the uppercase letters $U_k(x)$ represent the transformed form of this expression. The differential inverse transform of the function $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k \tag{2.2}$$

When Equation (2.1) and Equation (2.2) are considered together, the following expression is obtained:

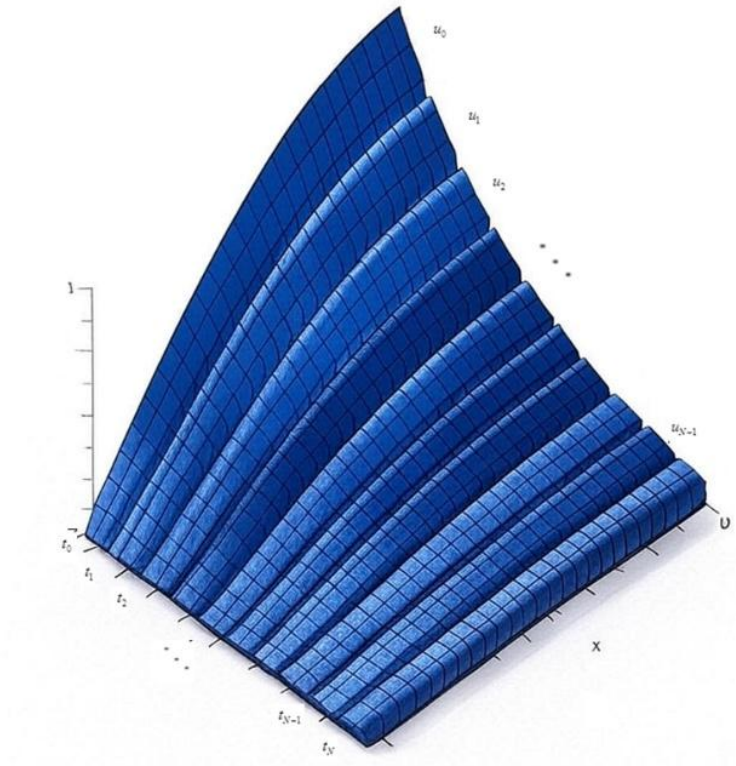
$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} u(x,t)\right]_{x=0} x^k \tag{2.3}$$

A review of the previous definitions clearly demonstrates that the RDTM approach was developed based on power series expansion. Based on this foundation and the operational rules in Table 1, the following recurrence relationship can be established:

Table 1. Reduced Differential Transformation

Function	**Transformation Equivalent**
$w(x,t)$	$W_k(t)=\frac{1}{k!}\left[\frac{\partial^k}{\partial x^k}w(x,t)\right]_{t=0}$
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(t)=U_k(t)\pm V_k(t)$
$w(x,t)=cu(x,t)$ $(c\in R)$	$W_k(t)=cU_k(t)$ $(c\in R)$
$w(x,t)=\frac{\partial u(x,t)}{\partial x}$	$W_k(t)=(k+1)U_{k+1}(t)$
$w(x,t)=\frac{\partial^r}{\partial x^r}u(x,t)$	$W_k(t)=\frac{(k+r)!}{k!}U_{k+r}(t)$
$w(x,t)=\frac{\partial^s}{\partial t^s}u(x,t)$	$W_k(t)=\frac{\partial^s}{\partial t^s}U_k(t)$
$w(x,t)=u(x,t)v(x,t)$	$W_k(t)=\sum_{r=0}^{k}V_r(t)U_{k-r}(t)=\sum_{k=0}^{k}U_r(t)V_{k-s}(t)$
$w(x,t)=x^m t^n$	$W_k(t)=\delta(k-m)t^n$
$w(x,t)=x^m t^n u(x,t)$	$W_k(t)=U_{k-m}(t)t^n$

We aim to investigate the solution of the differential equation (1.2) in the interval $\left[0,T\right]$. In this context, the relevant interval is divided into subregions N with equally spaced grid points defined as $\{t_0,t_{1,}\cdots,t_N\}$, as shown in Figure 1.



.

Figure 1. Approximation functions in each sub domain

In the formulation $t_0 = 0,\ t_N = T$ is taken as, and for each ~~$i = 0,1,2,\dots,N$~~. Here, the step width h is determined as,

$$t_i = t_0 + ih\ .\ h = \frac{T}{N} \tag{2.4}$$

After applying RDTM transformations to the function, the approximate solution on the first subinterval $[t_0, t_1]$ is defined as $u^0(x,t)$. This approximate solution is expressed by a Taylor polynomial of order *n*. developed around the point $t_0 = 0$.

$$\begin{aligned} u^0(x,t) &= U_0^0(x,t_0) + U_1^0(x,t_0)(t - t_0) \\ &+ U_2^0(x,t_0)(t-t_0)^2 + \cdots + U_n^0(x,t_0)(t-t_0)^n \end{aligned} \tag{2.5}$$

The initial condition of this polynomial is obtained by substituting it in (2.5),

$$u(x,t_0) = U_0^0(x,t_0) \tag{2.6}$$

When the value of $u^0(x,t)$ is calculated, this value is used as the initial condition for the approximate solution $u^0(x,t_1)$ in the interval $[t_1, t_2]$. The value of $u^0(x,t_1)$ is obtained as in equation (2.7).

$$\begin{aligned} u(x,t_1) &= u^0(x,t_1) \\ &= U_0^0(x,t_0) + U_1^0(x,t_0)(t_1 - t_0) + U_2^0(x,t_0)(t_1 - t_0)^2 + \\ &\cdots + U_n^0(x,t_0)(t_1 - t_0)^n \\ &= U_0^0 + U_1^0 h + U_2^0 h^2 + \cdots + U_n^0 h^n \\ &= \sum_{j=0}^{n} U_j^0 h^j \end{aligned} \tag{2.7}$$

and the approximate solution in this range is obtained using equation (2.8) given below.

$$\begin{aligned} u^1(x,t) &= U_0^1(x,t_1) + U_1^1(x,t_1)(t - t_1) \\ &+ U_2^1(x,t_1)(t - t_1)^2 + \cdots + U_n^1(x,t_1)(t - t_1)^n \end{aligned} \tag{2.8}$$

Following similar steps, $u^2(x,t)$ can be calculated as follows,

$$\begin{aligned} u(x,t_2) &= u^1(x,t_2) \\ &= U_0^1(x,t_1) + U_1^1(x,t_1)(t_2 - t_1) + U_2^1(x,t_1)(t_2 - t_2)^2 + \\ &\cdots + U_n^1(x,t_1)(t_2 - t_1)^n \\ &= U_0^1 + U_1^1 h + U_2^1 h^2 + \cdots + U_n^1 h^n \\ &= \sum_{j=0}^{n} U_j^1 h^j \end{aligned} \tag{2.9}$$

Following the same procedure, the approximate solution of the $u^i\left(x,t\right)$ function at grid point t_{i+1} can be obtained as follows.

$$\begin{aligned} u\left(x,t_{i+1}\right) &= u^i\left(x,t_{i+1}\right) \\ &= U_0^i\left(x,t_i\right) + U_1^i\left(x,t_i\right)\left(t_{i+1}-t_i\right) \\ &\quad + U_2^i\left(x,t_i\right)\left(t_{i+1}-t_i\right)^2 + \cdots + U_n^i\left(x,t_i\right)\left(t_{i+1}-t_i\right)^n \\ &= U_0^i + U_1^i h + U_2^i h^2 + \cdots + U_n^i h^n \\ &= \sum_{j=0}^{n} U_j^i h^j \end{aligned} \tag{2.10}$$

After determining the approximate solutions $u^1\left(x,t\right), u^2\left(x,t\right), ..., u^i\left(x,t\right)$ from the first step, the analytical solution of $u\left(x,t\right)$ is obtained as follows.

$$u\left(x,t\right) = \begin{cases} u^0\left(x,t\right), & 0 \le x \le T, \quad 0 < t \le t_1 \\ u^1\left(x,t\right), & 0 \le x \le T \quad t_1 < t \le t_2 \\ \vdots & \quad \vdots \quad\quad\quad \vdots \\ u^{N-1}\left(x,t\right), & 0 \le x \le T \quad t_{N-1} < t \le t_N \end{cases} \tag{2.11}$$

Increasing the value of N here can provide more accurate and precise approximate results.

2.2. Application Examples of the Proposed Method

In this section, the numerical results obtained are compared with analytical solutions to evaluate the effectiveness and advantages of the proposed method. The results are quite promising in demonstrating the accuracy and applicability of the method. To illustrate this, some example problems using the Klein-Gordon Equation are examined below. Furthermore, all computations were performed in Maple 13 on a 13th Gen Intel® Core™ i9-13900H system with 16 GB RAM under Windows 11.

Example 2.2.1

The homogeneous Klein-Gordon equation and the initial condition used are given in Eq. (2.12) and Eq.(2.13) as follows:

$$u_{tt} - u_{xx} + u = 0\,,\ t \ge 0 \tag{2.12}$$

$$u(x,0) = 0\,,\ u_t(x,0) = x \tag{2.13}$$

The analytical solution of Eq. (2.12) is $u(x,t) = x\sin(t)$ [38]. Now, let's obtain the approximate solution of Eq. (2.12) using the Fixed Grid Size Reduced Differential Transform Method (FGS-RDTM).

When $N=5$ is selected and $\left\{t_0 = 0,\ t_1 = \frac{1}{5},\ t_2 = \frac{2}{5},\ t_3 = \frac{3}{5},\ t_4 = \frac{4}{5},\ t_5 = 1\right\}$ values are used, the Reduced Differential Transform Method (RDTM) is applied to equation (2.12) with the help of the necessary transformation formulas given in Table 1 and Eq. (2.14) is obtained.

$$\frac{(k+2)!}{k!} U_{k+2}^0(x) = \frac{\partial^2}{\partial x^2} U_k^0(x) - U_k^0(x) \tag{2.14}$$

Here $U_k\left(x\right)$ is the reduced differential transform of the solution to be calculated along t of $u(x,t)$. Eq. (2.13) is obtained from the initial condition Eq. (2.15).

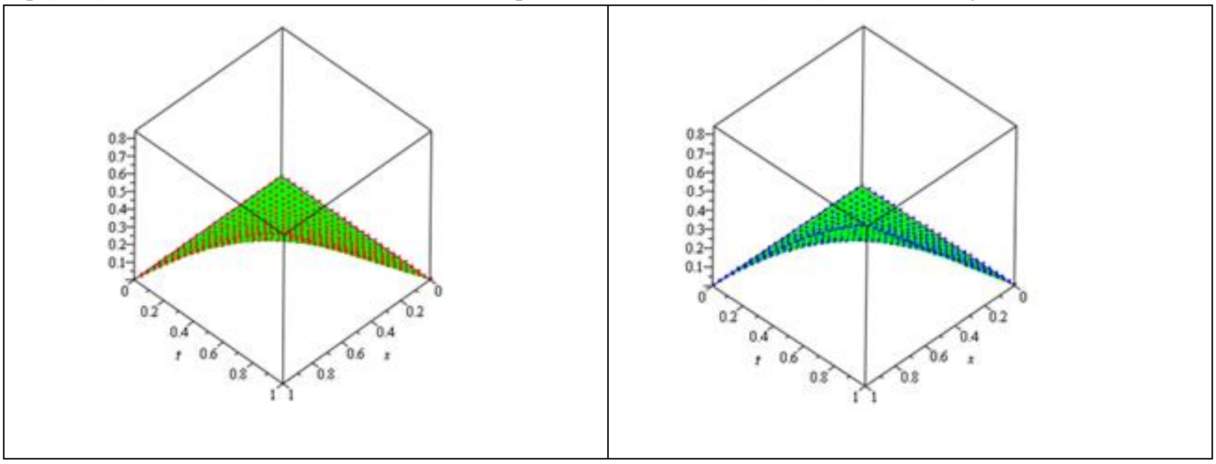
$$\begin{aligned} u(x,t_0) &= U_0^0(x,t_0) = 0 \\ U_1^0(x,t_0) &= x \end{aligned} \tag{2.15}$$

At this point, the initial conditions in (2.15) are substituted into equation (2.14) to obtain the values of $U_k^{N-1}(x)$. Then, using the values of $U_k^N(x)$ calculated for $N=5$, approximate solutions of $u^N(x,t)$ are presented in Table 2 below.

- **Table 2.** Approximate solutions for $u^N(x,t)$ for Example 2.2.1

$u^0(x,t)=xt-0.16666666666666666667xt^3+0.00833333333333333336xt^5-0.00019841269841269841270xt^7$ $+0.0000027557319223985890653xt^9-2.5052108385441718775.10^{-8}xt^{11}+1.6059043836821614599.10^{-10}xt^{13}$ $-7.6471637318198164760.10^{-13}xt^{15}+2.8114572543455207632.10^{-15}xt^{17}-8.2206352466243297168.10^{-18}xt^{19}$
$u^1(x,t)=-2.10^{-20}x+1.000000000000000001xt+6.1744069230464723146.10^{-21}xt^2-0.16666666666666666668xt^3$ $-2.69516535916638.10^{-22}xt^4+0.00833333333333333337xt^5+2.40649400817662.10^{-23}xt^6$ $-0.00019841269841269841274xt^7+2.71978296525.10^{-24}xt^8+0.0000027557319223985890421xt^9$ $+1.390046386.10^{-22}xt^{10}-2.5052108385442413965.10^{-8}xt^{11}+2.8975096471.10^{-21}xt^{12}$ $+1.6059043835818261454.10^{-10}xt^{13}+2.8680121565524026.10^{-20}xt^{14}-7.6471644013986451631.10^{-13}xt^{15}$ $+1.2563669091912641 6.10^{-19}xt^{16}+2.8112723166939901448.10^{-15}xt^{17}+2.05762963806646409.10^{-19}xt^{18}$ $-8.3834074744716279548.10^{-18}xt^{19}+8.3834074744716279548.10^{-20}xt^{20}$
$u^2(x,t)=-1.10^{-20}x+1.000000000000000001xt+3.7251584815835524 23.10^{-20}xt^2-0.16666666666666666668xt^3$ $+1.995124229256.10^{-21}xt^4+0.00833333333333333341xt^5+1.117099148831.10^{-21}xt^6$ $-0.00019841269841269841841xt^7+2.489612647617.10^{-20}xt^8+0.0000027557319223984990868xt^9$ $+2.7018606216.10^{-19}xt^{10}-2.5052108386117911780.10^{-8}xt^{11}+1.410528319171.10^{-18}xt^{12}$ $+1.6059043592319139891.10^{-10}xt^{13}+3.49940182334645363.10^{-18}xt^{14}-7.6472046531717465257.10^{-13}xt^{15}$ $+3.84783726558671207.10^{-18}xt^{16}+2.8086167146880293910.10^{-15}xt^{17}+1.5868368739716301 22.10^{-18}xt^{18}$ $-8.8522129317767617628.10^{-18}xt^{19}+1.6006330752322553430.10^{-19}xt^{20}$
$u^3(x,t)=-2.10^{-20}x+1.000000000000000002xt+4.50663708703188.10^{-21}xt^2-0.16666666666666666669xt^3$ $+1.75469630633.10^{-20}xt^4+0.00833333333333333332392xt^5+4.221594516192.10^{-19}xt^6$ $-0.00019841269841269992273xt^7+4.4116890012626.10^{-18}xt^8+0.0000027557319223879478429xt^9$ $+2.132976504833.10^{-17}xt^{10}-2.5052108421083424999.10^{-8}xt^{11}+4.965370327098.10^{-17}xt^{12}$ $+1.6059038086568437138.10^{-10}xt^{13}+5.50088728264223948.10^{-17}xt^{14}-7.6475939602114185587.10^{-13}xt^{15}$ $+2.70810314607516358.10^{-17}xt^{16}+2.7980566098392510819.10^{-15}xt^{17}+5.028531954183579508.10^{-18}xt^{18}$ $-9.5698149373402868773.10^{-18}xt^{19}+2.3208599092661840074.10^{-19}xt^{20}$
$u^4(x,t)=-6.10^{-20}x+1.000000000000000003xt+7.22445765868925.10^{-20}xt^2-0.16666666666666666705xt^3$ $+1.917837506378.10^{-18}xt^4+0.00833333333333333251632xt^5+2.7312010787962.10^{-17}xt^6$ $-0.00019841269841277178601xt^7+1.61040309789647.10^{-16}xt^8+0.0000027557319221067136984xt^9$ $+4.397339048207.10^{-16}xt^{10}-2.5052108937915148043.10^{-8}xt^{11}+5.7895429641183.10^{-16}xt^{12}$ $+1.6058993374780065848.10^{-10}xt^{13}+3.63596101443561018.10^{-16}xt^{14}-7.6493077685734119505.10^{-13}xt^{15}$ $+1.018964246322874929.10^{-16}xt^{16}+2.7733087276450690733.10^{-15}xt^{17}+1.086522419080611534.10^{-17}xt^{18}$ $-1.0445069943602371121.10^{-17}xt^{19}+2.9485613826146316926.10^{-19}xt^{20}$

Below are graphs showing the approximate solution of Eq. (2.12) found from FGS-RDTM and the comparison of the solution values found with the variation iteration method (VIM) with the analytical solution.



(a) **(b)**

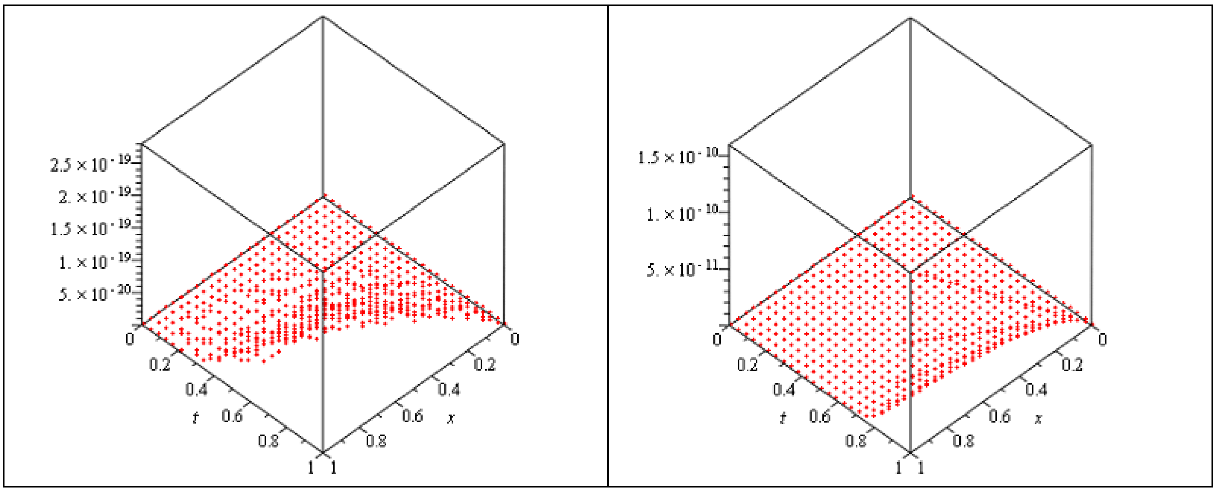
Figure 2. (a) Comparison of analytical solutions with FGS-RDTM. **(b)** Comparison of analytical solutions with VIM.

In this Figure2, for the value (a): The green area represents the analytical solution, the red dots represent the result obtained with the FGS-RDTM methods, (b): The green area represents the analytical solution, the blue dots represent the solution obtained with VIM.

In Table 3 below, the absolute error values of the results obtained with the FGS-RDTM and the VIM with the analytical solution are presented.

Table 3. Absolute Error Values Obtained by FGS-RDTM and VIM Compared to Analytical Solution.

	x	**Absolute Error Values FGS-RDTM**	**Absolute Error Values VIM**
	0.2	0.0	$2.6.10^{-20}$
	0.4	0.0	$5.3.10^{-20}$
t = 0.2	0.6	0.0	8.10^{-20}
	0.8	0.0	$1.1.10^{-19}$
	0.2	6.10^{-21}	$2.15377.10^{-16}$
	0.4	2.10^{-20}	$4.3076.10^{-16}$
	0.6	1.10^{-20}	$6.4613.10^{-16}$
t = 0.4	0.8	2.10^{-20}	$8.6151.10^{-16}$
	0.2	2.10^{-20}	$4.187663.10^{-14}$
	0.4	4.10^{-20}	$8.375327.10^{-14}$
	0.6	$1.2562990.10^{-20}$	$1.2562990.10^{-13}$
t = 0.6	0.8	8.10^{-20}	$1.6750654.10^{-13}$
	0.2	4.10^{-20}	$1.76034197.10^{-12}$
	0.4	8.10^{-20}	$3.52068393.10^{-12}$
	0.6	1.10^{-19}	$5.28102591.10^{-12}$
t = 0.8	0.8	$1.4.10^{-20}$	$7.04136787.10^{-12}$



(a) **(b)**

Figure 3. In the given graphs **(a):** FGS-RDTM Absolute Error Comparison **(b):** VIM Absolute Error Comparison

The absolute error graphs of Eq.(2.12) obtained with FGS-RDTM and VIM are shown in Figure 3.

Table 3 and Figure 3 compare the absolute error values of the approximate solutions obtained with the FGS-RDTM, and VIM for Example 2.2.1 at different values of x and t, compared to the analytical solution.

Analysis of the results reveals that the FGS-RDTM method produces extremely low error values across all time steps. In particular, the error values are nearly zero for $t=0.2$ and $t=0.4$, demonstrating high accuracy in the early time intervals.

In contrast, the error values obtained with the VIM method increase significantly as the time step increases. For example, at $t=0.8$ and $x=0.2$, the error for VIM is approximately $1.76034197.10^{-12}$, while at the same point, the error for FGS-RDTM is only 4.10^{-20}. This difference translates to approximately 8 orders of magnitude lower error.

The FGS-RDTM method produced stable and consistent results, even for x values that increase with time. This suggests that integrating the FGS-RDTM and constant grid spacing approach into the reduced differential transform method both increases accuracy and maintains numerical stability.

Example 2.2.2

Homogeneous Klein-Gordon equation, which is a type of hyperbolic equation,

$$u_{tt}-u_{xx}-u=0 \tag{2.16}$$

$$u(x,0)=1+\sin x,\ u_t(x,0)=0 \tag{2.17}$$

and analytical solution is $u(x,t)=\sin(x)+\cosh(t)$ [13].

Let's calculate the approximate solution of Eq. (2.16) with FGS-RDTM.

If the reduced differential transformation method is applied to Eq. (2.16) by choosing $\mathrm{N}=5$ and using Table 1, Eq (2.17) is obtained.

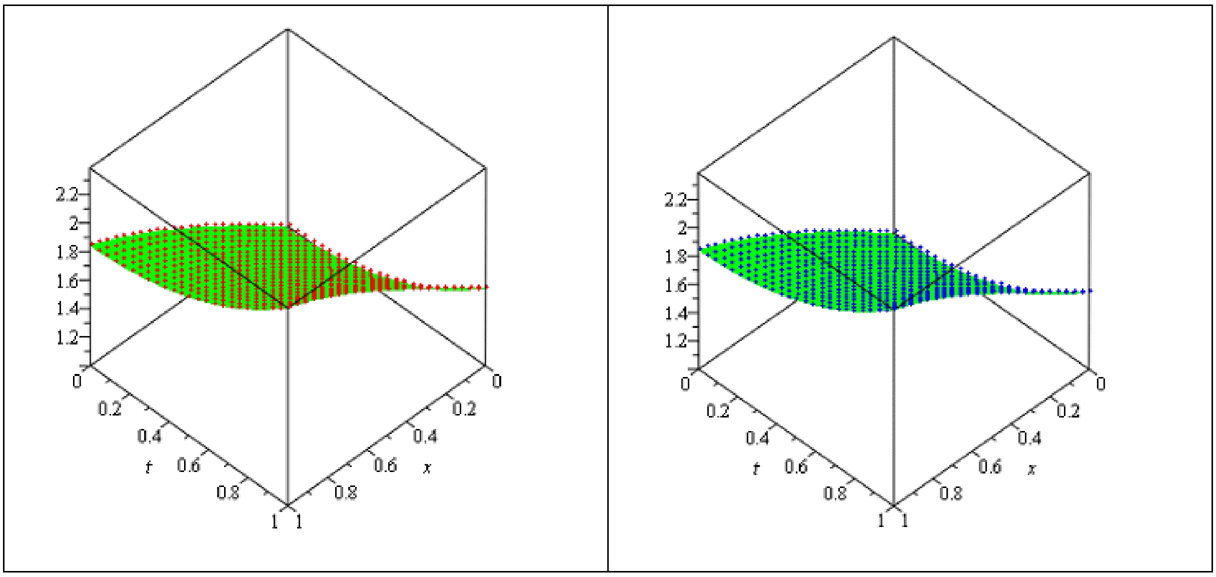
$$\frac{(k+2)!}{k!}U_{k+2}^{0}(x)=\frac{\partial^2}{\partial x^2}U_k^0(x)+U_k^0(x) \tag{2.18}$$

Here $U_k(x)$ is the reduced differential transform of the solution of $u(x,t)$ to be calculated along t. From the initial condition (2.17),

$$u(x,t_0)=U_0^0(x,t_0)=1+\sin(x)$$
$$U_1^0(x,t_0)=0 \tag{2.19}$$

From here, we can obtain the $U_k^{N-1}(x)$ values by substituting the initial conditions in (2.19) into (2.18).

Below are graphs showing the comparison of the analytical solution of equation (2.16) with the approximate solution found by FGS-RDTM and the results found by the variation iteration method.



(a) **(b)**

Figure 4. (a) Comparison of FGS-RDTM and the analytical solution, **(b)** Comparison of VIM and the analytical solution

where, for the value of N = 5, (a): The green area is the analytical solution, the red dots show the result obtained with the FGS-RDTM, (b): The green area is the analytical solution, the blue dots show the solution obtained with VIM.

The graph of absolute errors of equation (2.16) obtained with FGS-RDTM is given in Figure 5.

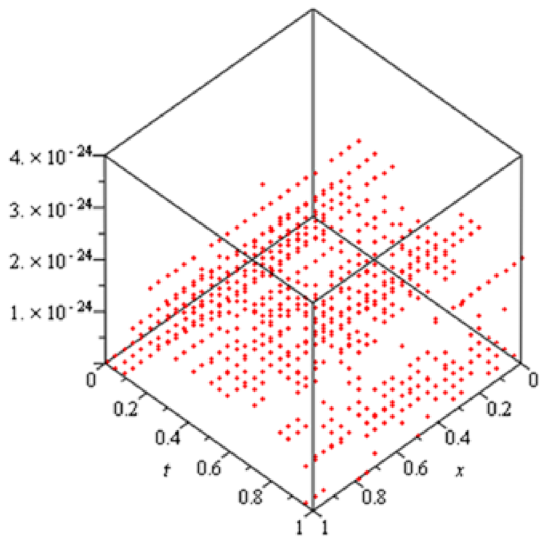


Figure 5. Absolute errors obtained with FGS-RDTM

The three-dimensional graph in the figure shows the absolute error distribution of the solution using the FGS-RDTM method compared to the analytical solution. Error values in the order of 10^{-24} clearly demonstrate that the developed method provides very high accuracy. Furthermore, the errors are observed to be distributed regularly and homogeneously in both the x and t directions. This demonstrates that the method not only provides precise results but also exhibits stable and consistent behavior throughout the solution space. These results demonstrate that the FGS-RDTM method is a powerful and effective approximate solution method that can be used with confidence in situations where an analytical solution cannot be found.

Example 2.2.3

Nonlinear Klein-Gordon equation and initial condition, which are linearly distorted by the $F(u) = -u^2$ term ,are given in Eq. (2.20) and Eq. (2.21) [39, 40].

$$u_{tt} - u_{xx} = -u^2 \tag{2.20}$$

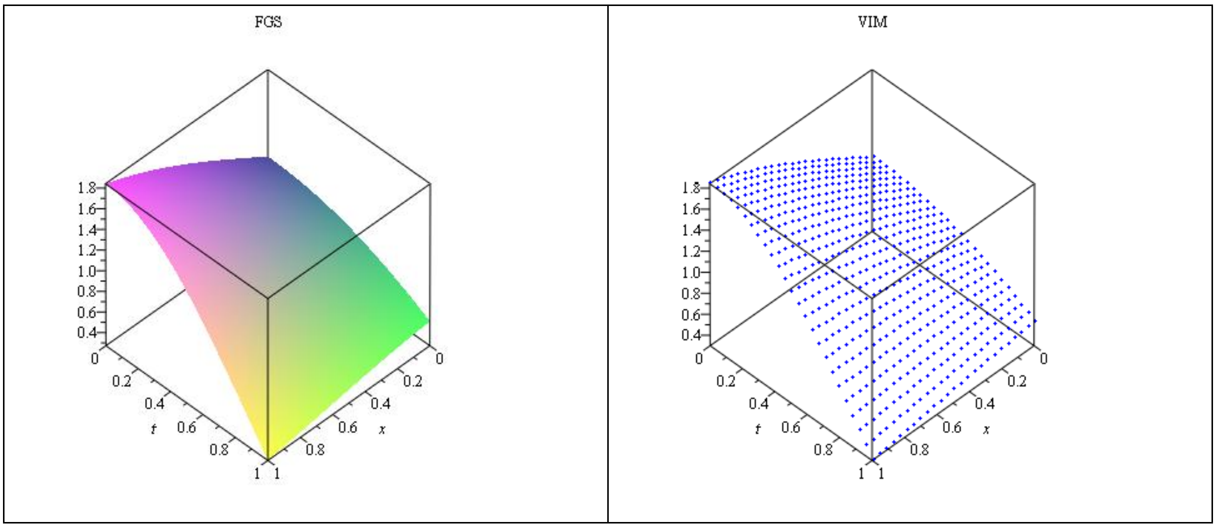
$$u(x,0) = 1 + \sin(x),\ u_t(x,0) = 0,\ t \geq 0 \tag{2.21}$$

Choosing $N = 5$ and dividing the interval $[0,1]$ into 5 equal parts, we obtain the t_i values. The reduced differential transform equivalent of equation (2.20) and the initial conditions (2.21) can be seen in Eq. (2.22).

$$\frac{(k+2)!}{k!}U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) - \sum_{s=0}^{k} U_{k-s}(x)U_s(x)$$
$$u(x,t_0) = U_0^0(x,t_0) = 1 + \sin(x),$$
$$U_1^0(x,t_0) = 0 \tag{2.22}$$

and exact solution of Eq. (2.20), $u(x,t) = x\sin(t)$

Substituting the obtained initial conditions into Eq. (2.22), we can obtain the $U_k^{N-1}(x)$ values. Then, for $N = 5$, the approximate solutions for $u^{N-1}(x,t)$ are found, respectively, using the calculated $U_k^{N-1}(x)$ values. The graphs showing the approximate results obtained after calculating the approximate solutions up to $u^5(x,t)$ are given in Figure 5 below.



(a) **(b)**

Figure 6. (a): FGS-RDTM (**b**): VIM

After calculating the approximate solution, the comparison of the obtained approximate results with ADM [39] and VIM [40] according to the selected x and t values is given in the table below.

- **Table 4.** Comparison Results

x	t = 0.2			t = 0.3		
	ADM	VIM	FGS-RDTM	ADM	VIM	FGS-RDTM
0.0	0.979999116	0.9800015775	0.9800000001	0.954990005	0.955017653	0.9549266388
0.2	1.166134875	1.16613805	1.166138093	1.125945576	1.125974851	1.125803615
0.4	1.343423788	1.343432104	1.343435074	1.287043874	1.287088824	1.286814817
0.6	1.505052082	1.505073495	1.505080634	1.432404521	1.432497282	1.432104538
0.8	1.644954933	1.64499754	1.645009692	1.557040327	1.557215916	1.556699252
1.0	1.75799845	1.758066925	1.758084239	1.656928567	1.657208637	1.656576413

The results presented in Table 4 reveal that the ADM, VIM, and proposed FGS-RDTM produce very similar and consistent approximate solutions for different x values at $t=0.2$ and $t=0.3$. The differences between the results obtained by all three methods are quite small, indicating that the FGS-RDTM exhibits competitive performance in terms of numerical stability. In particular, the FGS-RDTM method is in high agreement with ADM and VIM over the entire x range, producing more precise values at some points. This consistency supports the applicability and reliability of the method, thus indicating that FGS can be considered as an alternative to common methods in the literature.

3. RESULTS AND DISCUSSION

In this study, the proposed Fixed Grid Interval Reduced Differential Transform (FGS-RDTM) method demonstrated high success in obtaining approximate solutions of linear and nonlinear, homogeneous and nonhomogeneous partial differential equations. Numerical results obtained on applied example problems demonstrate the effectiveness of the method in terms of both accuracy and stability.

The three-dimensional absolute error distribution plot presented in Figure 4 illustrates the differences between the solution applied by combining the FGS-RDTM methods and the analytical solution. The magnitude of the errors remains on the order of 10^{-24}, demonstrating the method's exceptional accuracy. Furthermore, the regular and homogeneous structure of the error distribution in both the x and t directions demonstrates that the method not only achieves high accuracy but also exhibits numerically stable and consistent behavior throughout the entire solution space.

In addition, the approximate solutions obtained by the ADM, VIM, and proposed FGS-RDTM were comparatively evaluated at different x values for the instants $t=0.2$ and $t=0.3$. The results in Table 4 show that all methods produce similar outputs; however, the FGS method stands out by providing more accurate results at some points. Compared to the ADM and VIM methods, the solutions produced by FGS-RDTM have significantly lower numerical deviations and are consistent across the entire range. This demonstrates that the proposed method not only has competitive accuracy but also provides robust and reliable solutions under different initial and boundary conditions.

4. CONCLUSIONS

In this study, a new algorithm with a fixed grid spacing, developed based on the reduced differential transform method, is proposed and its success in approximate solutions of partial differential equations is evaluated. By dividing the solution range into equal subregions and applying iteratively to each subregion, the developed FGS-RDTM method produces highly accurate solutions with very small error values. Furthermore, the ability to increase the order according to the desired error tolerance thanks to the recurrence relation, which only includes derivative terms, provides flexibility and ease of application.

One of the most important advantages of the method is its easy applicability to many types of PDEs frequently encountered in the literature, including linear/non-linear and homogeneous/inhomogeneous. Analyses supported by three-dimensional error distribution plots clearly demonstrated the method's superior performance in terms of both accuracy and stability. Furthermore, comparisons with common methods such as ADM and VIM revealed that the FGS method is at least as effective and reliable as an alternative.

In conclusion, the FGS-RDTM method can be used as a powerful and effective approximate solution technique for problems where analytical solutions cannot be obtained. Future work is recommended to apply the method to more challenging problem groups, such as those with more complex boundary conditions, time-dependent coefficients, or stochastic differential equations.

Declaration of Ethical Standards

The authors confirm that the appropriate ethics review has been followed

Credit Authorship Contribution Statement

Sema Servi and Galip Oturanc conducted the literature review of the manuscript and the design of the proposed method together. It also contributed equally to obtaining the results of the proposed method and interpreting the results. Sema Servi and Galip Oturanc read and approved the final manuscript.

Declaration of Competing Interest

The authors declare no conflict of interest.

Funding / Acknowledgements

All authors declare that there is no funding for this work.

Data Availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

References

[1] İ. Ateş and A. Yıldırım, "Comparison between variational iteration method and homotopy perturbation method for linear and nonlinear partial differential equations with the nonhomogeneous initial conditions," *Numerical Methods for Partial Differential Equations,* vol. 26, no. 6, pp. 1581-1593, 2010.

[2] M. A. Shihab, W. M. Taha, R. A. Hameed, A. Jameel, and S. Ibrahim, "Implementation of variational iteration method for various types of linear and nonlinear partial differential equations," *International Journal of Electrical and Computer Engineering,* vol. 13, no. 2, pp. 2131-2141, 2023.

[3] D. Ganji, A. Rezania, and A. Ghorbali, "On homotopy perturbation and variational iteration methods for linear and nonlinear heat transfer equations," *Journal of Algorithms & Computational Technology,* vol. 2, no. 3, pp. 371-389, 2008.

[4] Q. Gao, J. Li, W. Wang, X. Wang, H.-J. Hwang, and Y. H. Lee, "Homotopy analysis method for linear and nonlinear stochastic problems," *Probabilistic Engineering Mechanics,* vol. 79, p. 103732, 2025.

[5] P. Pathak, A. K. Barnwal, N. Sriwastav, R. Singh, and M. Singh, "An algorithm based on homotopy perturbation theory and its mathematical analysis for singular nonlinear system of boundary value problems," *Mathematical Methods in the Applied Sciences,* vol. 48, no. 7, pp. 7745-7766, 2025.

[6] N. Sharif, M. S. Alam, and H. U. Molla, "Dynamics of nonlinear pendulum equations: Modified homotopy perturbation method," *Journal of Low Frequency Noise, Vibration and Active Control,* p. 14613484251320219, 2025.

[7] A. Kurnaz, G. Oturanç, and M. E. Kiris, "n-Dimensional differential transformation method for solving PDEs," *International Journal of Computer Mathematics,* vol. 82, no. 3, pp. 369-380, 2005.

[8] İ. Çelik and D. Eskin, "Differential transformation method for nonlinear differential-difference equations," *Ordu Üniversitesi Bilim ve Teknoloji Dergisi,* vol. 5, no. 1, pp. 37-50, 2016.

[9] Y. Keskin and G. Oturanc, "Reduced differential transform method for partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 10, no. 6, pp. 741-750, 2009.

[10] S. Servi, Y. Keskin, and G. Oturanc, "Improved approximation algorithm for solution of nonlinear partial differential equations," *arXiv preprint arXiv:1512.08717,* 2015.

[11] Y. Keskin and G. Oturanc, "Reduced differential transform method for generalized KdV equations," *Mathematical and Computational applications,* vol. 15, no. 3, pp. 382-393, 2010.

[12] Y. Keskin and G. Oturanç, "Application of reduced differential transformation method for solving gas dynamics equation," *International Journal of Contemporary Mathematical Sciences,* vol. 22, no. 22, pp. 1091-1096, 2010.

[13] Y. Keskin, S. Servi, and G. Oturanç, "Reduced differential transform method for solving Klein Gordon equations," in *Proceedings of the World Congress on engineering*, 2011, vol. 1: WCE, p. 2011.

[14] Y. Keskin and G. Oturanc, "Numerical solution of regularized long wave equation by reduced differential transform method," *Applied Mathematical Sciences,* vol. 4, no. 25, pp. 1221-1231, 2010.

[15] S. Servi, Y. Keskin, and G. Oturanç, "Reduced differential transform method for improved Boussinesq equation," in *AIP Conference Proceedings*, 2015, vol. 1648, no. 1: AIP

Publishing LLC, p. 370012.
[16] R. Abazari and B. Soltanalizadeh, "Reduced differential transform method and its application on Kawahara equations," *Thai Journal of Mathematics,* vol. 11, no. 1, pp. 199-216, 2013.
[17] K. K. Chaudhari and C. S. Sutar, "Reduced differential transform method for thermoelastic problem in hyperbolic heat conduction domain," 2021.
[18] V. K. Srivastava, N. Mishra, S. Kumar, B. K. Singh, and M. K. Awasthi, "Reduced differential transform method for solving (1+ n)–Dimensional Burgers' equation," *Egyptian journal of Basic and Applied sciences,* vol. 1, no. 2, pp. 115-119, 2014.
[19] A. R. Appadu and A. S. Kelil, "Solution of 3D linearized KdV equation using reduced differential transform method," in *AIP Conference Proceedings*, 2022, vol. 2425, no. 1: AIP Publishing LLC, p. 020016.
[20] R. Oderinu and A. Oyewumi, "The Aboodh reduced differential transform method for the Hirota-Satsuma coupled KDV and MKDV equations," *J. Math. Comput. Sci.,* vol. 12, p. Article ID 135, 2022.
[21] S. Servi, "Improved reduced differential transform method for solution of linear and nonlinear partial differential equations," PhD Thesis, Selcuk University, Institute of Science, 2016.
[22] N. Bildik and A. Konuralp, "The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 7, no. 1, p. 65, 2006.
[23] G. Adomian, "Solutions of nonlinear PDE," *Applied Mathematics Letters,* vol. 11, no. 3, pp. 121-123, 1998.
[24] D. A. Maturi and H. M. Malaikah, "The adomian decomposition method for solving nonlinear partial differential equation using maple," *Advances in Pure Mathematics,* vol. 11, no. 6, pp. 595-603, 2021.
[25] S. Chakraverty, R. M. Jena, and S. K. Jena, *Computational fractional dynamical systems: fractional differential equations and applications*. John Wiley & Sons, 2022.
[26] U. Arshad, "An Elzaki Transform Decomposition Algorithm Applied to a class of nonlinear DE," *Journal of Natural Sciences Research,* 2015.
[27] I. Sumiati, E. Rusyaman, S. Sukono, and A. Bon, "A review of adomian decomposition method and applied to deferential equations," in *Proceedings of the International Conference on Industrial Engineering and Operations Management, Pilsen, Czech Republic, July*, 2019, pp. 23-26.
[28] G. Nhawu, P. Mafuta, and J. Mushanyu, "The Adomian decomposition method for numerical solution of first-order differential equations," *J. Math. Comput. Sci.,* vol. 6, no. 3, pp. 307-314, 2016.
[29] A. Sadeghinia and P. Kumar, "One Solution of Multi-term Fractional Differential Equations by Adomian Decomposition Method: Scientific Explanation," *Current Topics on Mathematics and Computer Science Vol. 6,* pp. 120-130, 2021.
[30] Z. A. Hasan and A.-S. J. Al-Saif, "A New Approach to Solving Two-Dimensional Unsteady Incompressible Navier-Stokes Equations," *Journal of Applied Mathematics and Physics,* vol. 10, no. 10, pp. 3218-3239, 2022.
[31] S. Fadugba *et al.*, "Solving the Black–Scholes European options model using the reduced differential transform method with powered modified log-payoff function," *Partial Differential Equations in Applied Mathematics,* vol. 13, p. 101087, 2025.
[32] O. Acan and Y. Keskin, "A new technique of Laplace Pade reduced differential transform method for (1+ 3) dimensional wave equations," *New Trends in Mathematical Sciences,* vol. 5, no. 1, pp. 164-171, 2017.
[33] S. Servı and G. Oturanç, "Reduced Differential Transform Approach Using Fixed Grid Size for Solving Newell–Whitehead–Segel (NWS) Equation," in *The International Conference on Artificial Intelligence and Applied Mathematics in Engineering*, 2020: Springer, pp. 304-314.
[34] S. Servi, Y. Keskin, and G. Oturanç, "Reduced Differential Transform Method With Fixed Grid Size For Solving Telegraph Equations," *Book of Abstracts Computational Methods in Applied Mathematics (CMAM-7) July 31-August 6, 2016 University of Jyväskylä, Finland,* p. 129, 2016.
[35] S. Servi, Y. Keskin, and G. Oturanç, "Numerical Solution Of Wave Equations By Reduced Differential Transform Method With Fixed Grid Size," *Book of Abstracts Computational Methods in Applied Mathematics (CMAM-7) July 31-August 6, 2016 University of Jyväskylä, Finland,* p. 128, 2016.
[36] S. SERVİ, "Reduced differential transform method with fixed grid size for solving Gaousat problem," *Innovations and Technologies in Engineering,* p. 83, 2022.
[37] M.-J. Jang, C.-L. Chen, and Y.-C. Liy, "On solving the initial-value problems using the differential transformation method," *Applied Mathematics and Computation,* vol. 115, no. 2-3, pp. 145-160, 2000.
[38] A.-M. Wazwaz, *Partial differential equations and solitary waves theory*. Springer Science & Business Media, 2010.
[39] S. M. El-Sayed, "The decomposition method for studying the Klein–Gordon equation," *Chaos, Solitons & Fractals,* vol. 18, no. 5, pp. 1025-1030, 2003.
[40] E. Yusufoğlu, "The variational iteration method for studying the Klein–Gordon equation," *Applied Mathematics Letters,* vol. 21, no. 7, pp. 669-674, 2008.